A Category Theoretic Treatment of Robot Hybrid Dynamics with Applications to Reactive Motion Planning and Beyond

Jared Culbertson[1] **Paul Gustafson**[2] Dan Koditschek[3]
Peter Stiller[4]

[1]Air Force Research Laboratory

[2]Wright State University

[3]University of Pennsylvania [4]Texas A&M University

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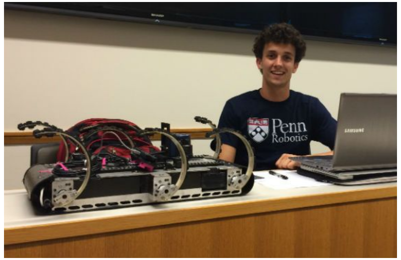
Can we make behaviors modular?

Robot Whisperer:

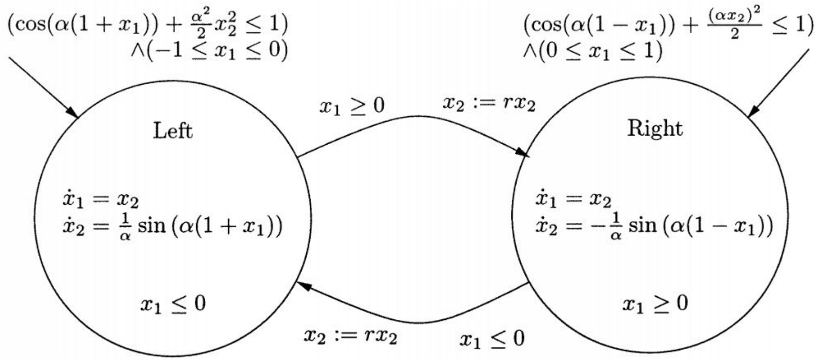


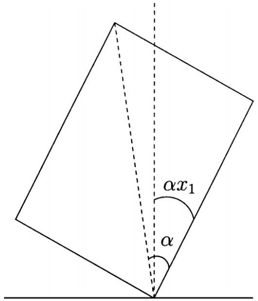
English	Type Theory
True	$\mathbf{1}$
False	$\mathbf{0}$
A and B	$A \times B$
A or B	$A + B$
If A then B	$A \to B$
A if and only if B	$(A \to B) \times (B \to A)$
Not A	$A \to \mathbf{0}$

Hybrid systems

A **hybrid system** H consists of

- a directed graph $G = (V, E, \mathfrak{s}, \mathfrak{t})$;
- for each **mode** $v \in V$,
  - an **ambient smooth system** (M_v, X_v)
  - an **active set** $I_v \subset M_v$
  - a **flow set** $F_v \subset I_v$
- for each **reset** $e \in E$, a **guard set** $Z_e \subset I_{\mathfrak{s}(e)}$ and an associated **reset map** $r_e \colon Z_e \to I_{\mathfrak{t}(e)}$.



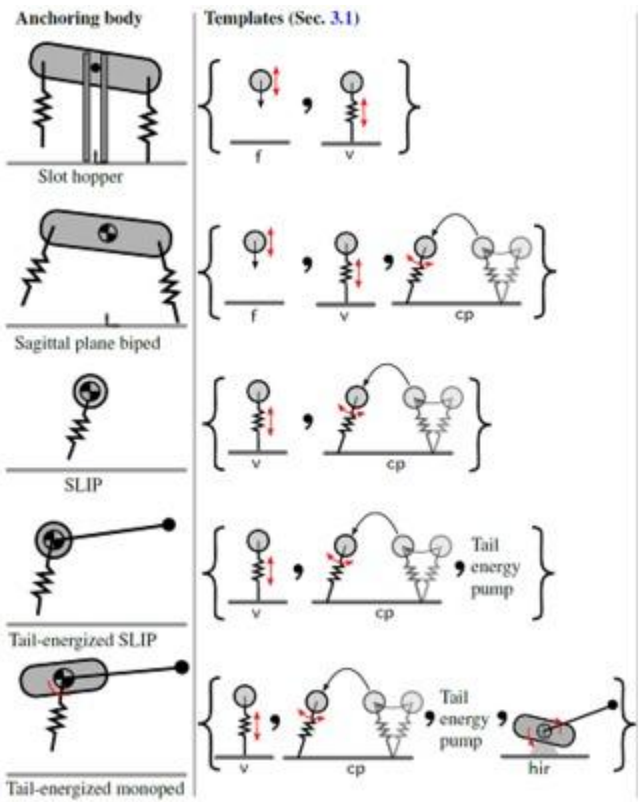


Morphisms: hybrid semiconjugacies

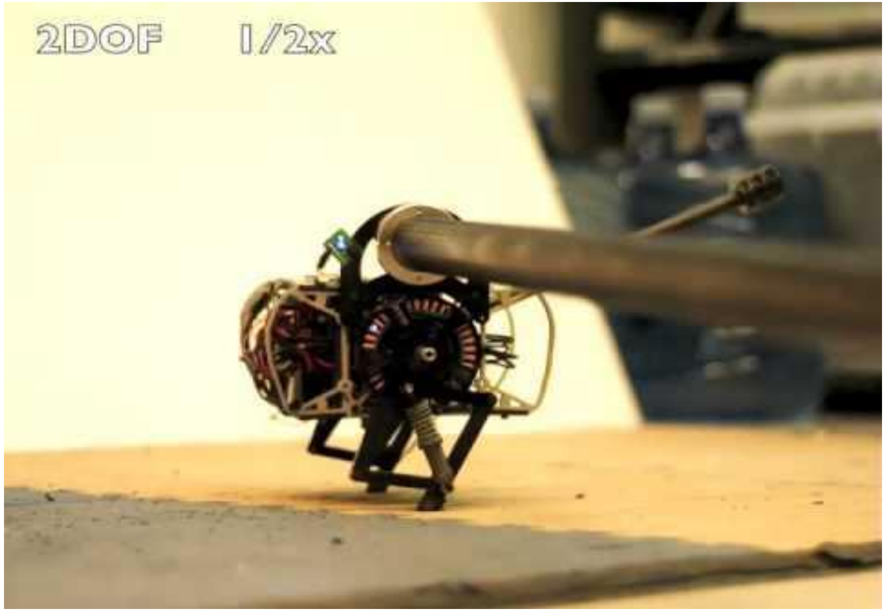
- “execution-preserving maps”

Cf. Lerman. “A category of hybrid systems.” arXiv:1612.01950.

Image source: Lygeros et al., ”Dynamical properties of hybrid automata.” IEEE Transactions on automatic control, 2003.



Templates and anchors

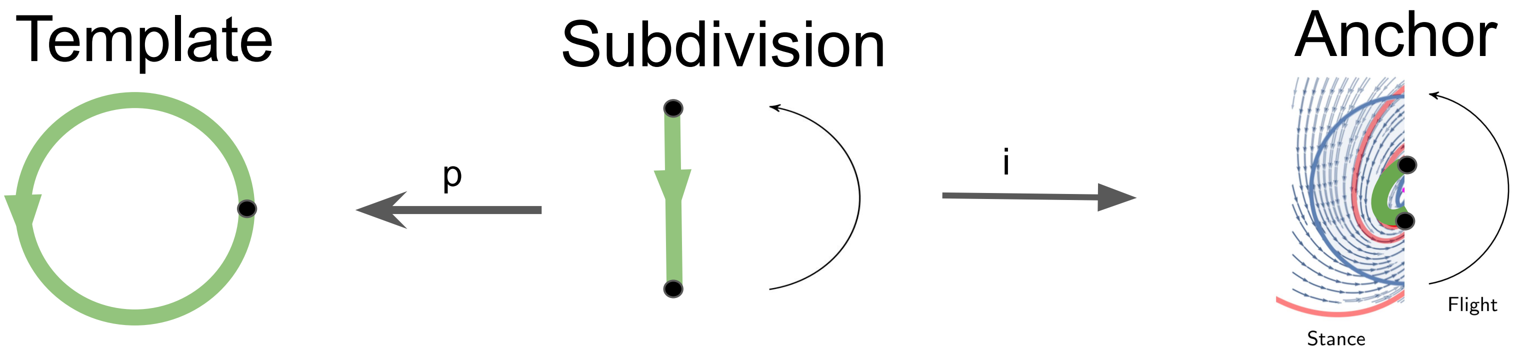


De, Avik, and Daniel E. Koditschek. "Parallel composition of templates for tail-energized planar hopping." 2015 IEEE International Conference on Robotics and Automation (ICRA). IEEE, 2015.

Anchoring a limit cycle in a vertical hopper

A **template-anchor pair** is a span $T \xleftarrow{p} S \xrightarrow{i} A$ such that

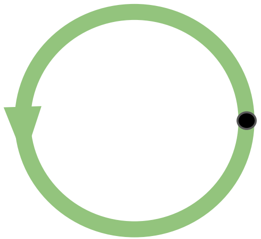
- p is a hybrid subdivision;
- i is a hybrid embedding;
- $i(S)$ is attracting in A.

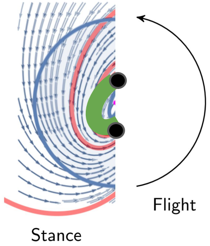


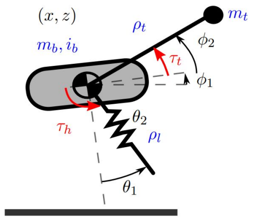
Hierarchical composition

Theorem (CGKS). Template-anchor pairs are weakly associatively composable.

$$
\begin{tikzcd}
 & & S = S_1 \times_{A_1} S_2 \arrow[dl, two heads] \arrow[dr, hook] \arrow[dd, phantom, "\vee"] \arrow[ddll, bend right, two heads, "p"'] \arrow[ddrr, bend left, hook, "i"] & & \\
 & S_1 \arrow[dl, two heads] \arrow[dr, hook] & & S_2 \arrow[dl, two heads] \arrow[dr, hook] & \\
T_1 & & A_1 & & A_2
\end{tikzcd}
$$

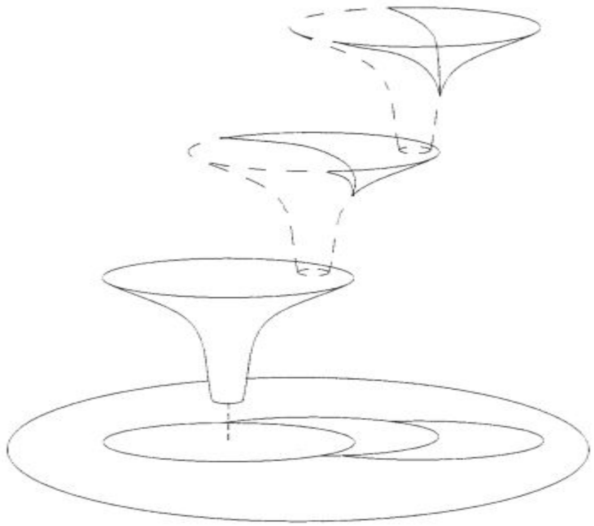






Tailed monoped

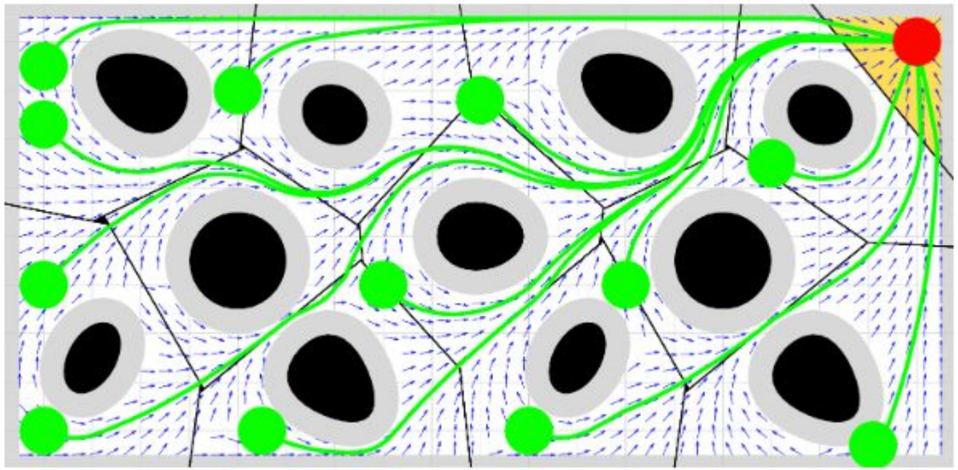
Sequential composition



Goal: define a class of “funnel-like” hybrid systems closed under sequentially composition

Burridge, Robert R., Alfred A. Rizzi, and Daniel E. Koditschek. "Sequential composition of dynamically dexterous robot behaviors." *The International Journal of Robotics Research* 18.6 (1999): 534-555.

A “navigate-to-goal” funnel



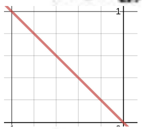
Theorem 3. *The piecewise continuously differentiable “move-to-projected-goal” law in (11) leaves the robot’s free space $\mathcal{F}$ (1) positively invariant; and if Assumption 2 holds, then its unique continuously differentiable flow, starting at almost[1] any configuration $\mathrm{x} \in \mathcal{F}$, asymptotically reaches the goal location x^*, while strictly decreasing the squared Euclidean distance to the goal, $\|\mathrm{x} - \mathrm{x}^*\|^2$, along the way.*

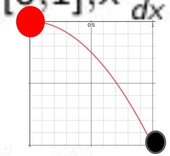
Arslan, Omur, and Daniel E. Koditschek. "Sensor-based reactive navigation in unknown convex sphere worlds." *The International Journal of Robotics Research* (2019).

How to define ``funnel-like" systems?

- **Problem:** the naive measure-theoretic and topologically notions of "almost all" are incompatible with fully general sequential composition
- Example:

$$H = \left([-1,0], \tfrac{d}{dx}\right) \xrightarrow[0 \mapsto 0]{} (\{0\},0) \qquad K = \left([0,1], x\tfrac{d}{dx}\right) \xrightarrow[1 \mapsto 1]{} (\{1\},0)$$





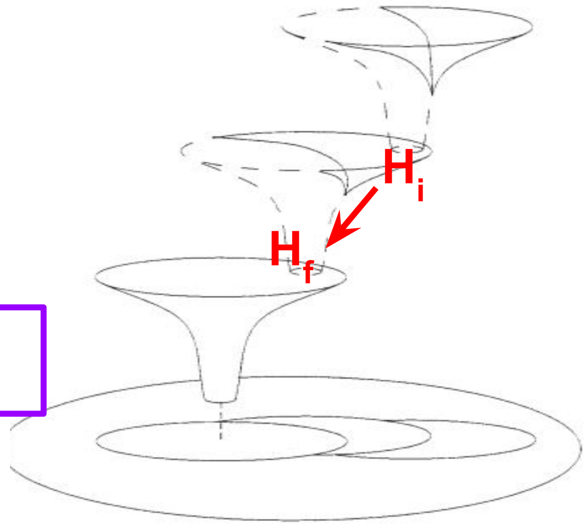
- Is there a notion of "generalized execution" compatible with sequential composition?

Directed systems

A **directed hybrid system** $H \colon H_i \rightsquigarrow H_f$ is a tuple (H, η_i, η_f) consisting of

- a metric hybrid system H,
- embeddings $\eta_i \colon H_i \to H$ and
- a hybrid embedding $\eta_f \colon H_f \to H$ such that each component $(\eta_f)_v$ is a diffeomorphism, and $G(H_f)$ is a sink in $G(H)$

such that for all $\varepsilon, T > 0$ and $x \in H$, there exists an (ε, T)-**chain** from x to some $y \in H_f$.



Composable!

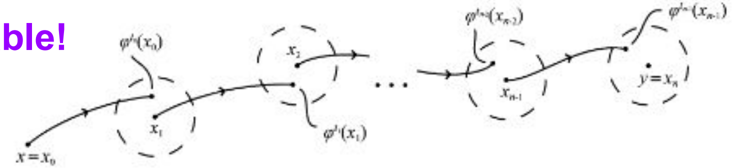
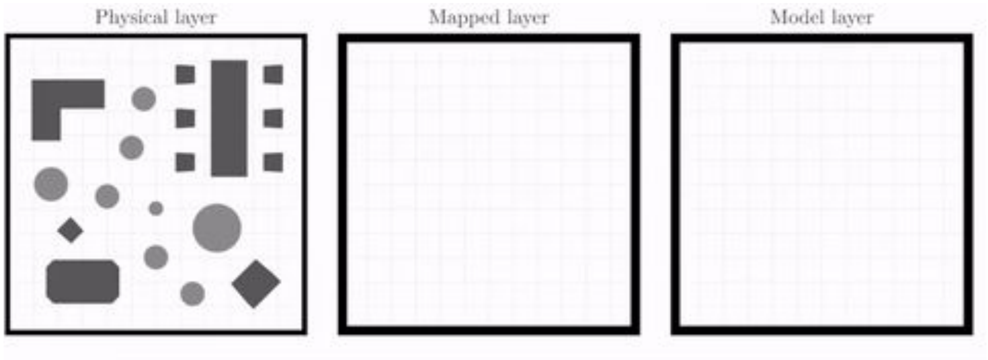


Image source: Alongi and Nelson, *Recurrence and Topology*. AMS, 2007.

A double category of hybrid systems



V. Vasilopoulos, D.E. Koditschek (2018). Reactive Navigation in Partially Known Non-Convex Environments. In WAFR 2018.

$$
\begin{tikzcd}
H_i \arrow[r, "\alpha_i"] \arrow[d, squiggly, "H"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & K_i \arrow[d, squiggly, "K"] \\
H_f \arrow[r, "\alpha_f"'] & K_f
\end{tikzcd}
$$

Challenges for Topological Complexity Theorists

- Effective versions of TC for given affordances
  - Which paths can we anchor in a realistic system?
  - Physically grounded TC:
    - Path -> directed system
    - End point -> (steady-state) behavior

Thanks!

arXiv:1911.01267